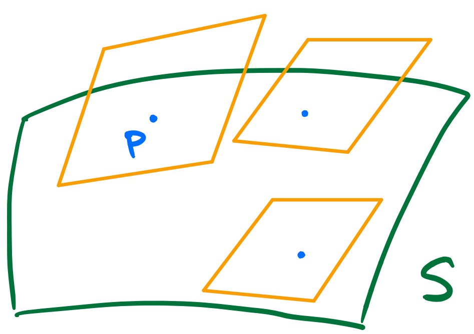
§ First Fundamental Form (do Carmo § 2.5)

Recall that we have defined at each p on a surface S the tangent plane T_pS, which is a 2-dimensional subspace of $\mathbb{R}^3$. By putting all these tangent planes together, we get

tangent bundle of S : $\boxed{TS := \{(p, v) : p \in S,\ v \in T_pS\}}$

Note: We can think of TS as a 2-parameter family of 2-dimensional vector spaces (i.e. T_pS) parametrized by points p on S.



"disjoint union"

$$TS = \coprod_{p \in S} T_pS$$

Since each T_pS is a subspace of $\mathbb{R}^3$, it inherits the inner product from $\mathbb{R}^3$ as well. Therefore, we have the following:

Def$^{\underline{n}}$: The first fundamental form (1^{st} f.f.) of a surface at a point $p \in S$ is a positive definite, symmetric bilinear form (i.e. an inner product) defined on T_pS by

$$g_p : T_pS \times T_pS \longrightarrow \mathbb{R}$$

$$\boxed{g_p(u, v) := \langle u, v \rangle_{\mathbb{R}^3}}$$

standard inner product in $\mathbb{R}^3$

Note: TS is then a smooth family of inner product spaces parametrized by S.

We can express the 1^{st} f.f. locally as 2×2 matrices (g_{ij}) using coordinate systems as follows:

Given a parametrization $X(u_1, u_2) : \mathcal{U} \longrightarrow S$,

$$T_pS = \text{Span}\left\{ \frac{\partial X}{\partial u_1}, \frac{\partial X}{\partial u_2} \right\}$$

we can express g_p by a matrix as

$$(g_{ij}) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \quad \text{where} \quad g_{ij} = \left\langle \frac{\partial X}{\partial u_i}, \frac{\partial X}{\partial u_j} \right\rangle \quad (i,j = 1,2)$$

2×2 symmetric matrix

Therefore, if $u = a\frac{\partial}{\partial u_1} + b\frac{\partial}{\partial u_2}$

$v = c\frac{\partial}{\partial u_1} + d\frac{\partial}{\partial u_2}$ where $\frac{\partial}{\partial u_i} = \frac{\partial X}{\partial u_i}$

then

$$g(u,v) = (a \quad b)\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix}.$$

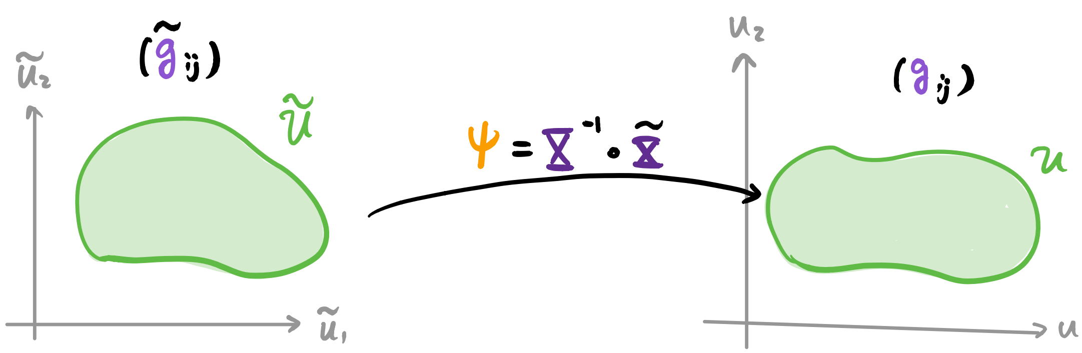
<u>Question</u>: How does the matrix (g_{ij}) transform when we change local coordinates?

<u>Lemma</u>: (Transformation law for (g_{ij}))

Suppose (g_{ij}) and $(\tilde{g}_{ij})$ are the 1st f.f. expressed in local coordinates $X(u_1,u_2): \mathcal{U} \to S$ and $\tilde{X}(\tilde{u}_1,\tilde{u}_2): \tilde{\mathcal{U}} \to S$ respectively. Then,

$$\boxed{(\tilde{g}_{ij}) = (D\psi)^T (g_{ij})(D\psi)}$$

where $\psi = X^{-1} \circ \tilde{X} : \tilde{\mathcal{U}} \to \mathcal{U}$ is the transition map.



Proof: First of all,

$$(g_{ij}) = \begin{pmatrix} \left\langle \frac{\partial X}{\partial u_1}, \frac{\partial X}{\partial u_1} \right\rangle & \left\langle \frac{\partial X}{\partial u_1}, \frac{\partial X}{\partial u_2} \right\rangle \\ \left\langle \frac{\partial X}{\partial u_2}, \frac{\partial X}{\partial u_1} \right\rangle & \left\langle \frac{\partial X}{\partial u_2}, \frac{\partial X}{\partial u_2} \right\rangle \end{pmatrix} \quad 2\times 2$$

$$= \underbrace{\begin{pmatrix} — \frac{\partial X}{\partial u_1} — \\ — \frac{\partial X}{\partial u_2} — \end{pmatrix}}_{2\times 3} \underbrace{\begin{pmatrix} | & | \\ \frac{\partial X}{\partial u_1} & \frac{\partial X}{\partial u_2} \\ | & | \end{pmatrix}}_{3\times 2} = (DX)^T(DX)$$

On the other hand,

$$(\tilde{g}_{ij}) = (D\tilde{X})^T(D\tilde{X})$$

$$= (D\psi)^T(DX)^T(DX)(D\psi) \qquad (\because \tilde{X} = X \circ \psi)$$

$$= (D\psi)^T(g_{ij})(D\psi)$$

$\square$

Corollary: $\sqrt{\det(\tilde{g}_{ij})} = \sqrt{\det(g_{ij})}\ |\mathrm{Jac}\,\psi|$

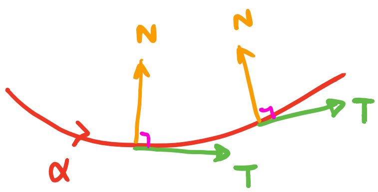
Note: $\sqrt{\det(g_{ij})} = \left\| \frac{\partial X}{\partial u_1} \times \frac{\partial X}{\partial u_2} \right\|$

$$\Rightarrow \int_S f \underset{\text{locally}}{=} \int_U f \underbrace{\sqrt{\det(g_{ij})}\, du_1 du_2}_{dA\ :\ \text{area form.}}$$

§ Gauss map & Second Fundamental Form (do Carmo § 3.2)

We now study the extrinsic geometry of surfaces and define various notions of curvatures for surfaces.

Recall: (Plane curves)



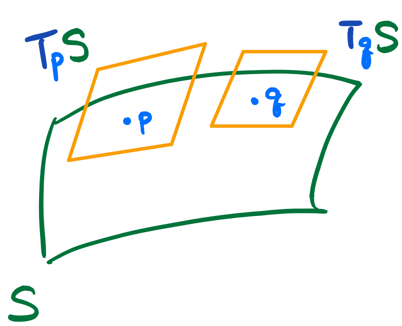
Frenet eq$^{\underline{n}}$: $\begin{pmatrix} T \\ N \end{pmatrix}' = \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix} \begin{pmatrix} T \\ N \end{pmatrix}$

$\Downarrow$

$$\left[\begin{array}{ll} \text{Curvature } k & = \text{rate of change of } T \\ & = -(\text{rate of change of } N) \end{array}\right]$$

Now, for an (orientable) surface S, we consider its

tangent bundle TS "=" a family of tangent planes T_pS



HOPE:

"Curvature of S" $=$ rate of change of T_pS

$= -$ rate of change of unit normal N_p

Note: In $\mathbb{R}^3$, a 2-dim'l subspace P is determined uniquely (up to a sign) by its unit normal $N \perp P$.

Def$^{\underline{n}}$: Let $S \subseteq \mathbb{R}^3$ be an orientable surface, oriented by a global unit normal vector field called

$$N : S \longrightarrow \mathbb{S}^2 \qquad \text{Gauss map}$$

unit sphere in $\mathbb{R}^3$

The Gauss map N is a smooth map from S to $\mathbb{S}^2$

$\Rightarrow$ we can consider its differential at any $p \in S$

$$dN_p : T_pS \xrightarrow{\text{linear}} T_{N(p)}\mathbb{S}^2 \cong N(p)^{\perp} = T_pS$$

Def$^{\underline{n}}$: The shape operator / Weingarten map (at p) is the linear operator on T_pS defined by

$$\mathcal{S} = -dN_p : T_pS \xrightarrow{\text{linear}} T_pS$$

Def$^{\underline{n}}$:

$$\boxed{H := \operatorname{tr} \mathcal{S}} \quad \text{mean curvature}$$

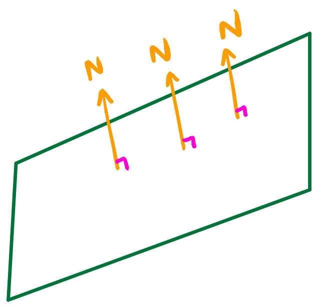
$$\boxed{K := \det \mathcal{S}} \quad \text{Gauss curvature}$$

Effect of orientation:

$$N \rightsquigarrow -N \Rightarrow \mathcal{S} \rightsquigarrow -\mathcal{S} \Rightarrow \underbrace{\begin{matrix} H \rightsquigarrow -H \\ K \rightsquigarrow K \end{matrix}}_{\text{unchanged !!}}$$

Examples:

(1) Planes



$N \equiv$ const. vector

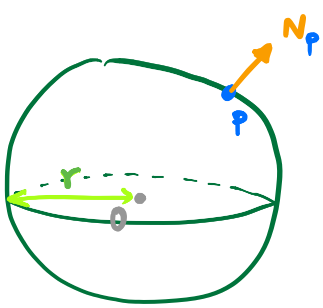
$$\Rightarrow \quad S = -dN = 0 \qquad \text{i.e.} \quad S = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad \boxed{\begin{array}{l} H \equiv 0 \\ K \equiv 0 \end{array}}$$

"flat"

(2) Spheres

$$S = \mathbb{S}^2(r) = \{ p \in \mathbb{R}^3 : \|p\| = r \}$$



$S = \mathbb{S}^2(r)$

$$N(p) = \frac{p}{\|p\|} = \frac{1}{r} p$$

$$\Rightarrow \quad S = -dN = -\frac{1}{r} \mathrm{Id}, \quad \text{i.e.} \quad S = \begin{pmatrix} -\frac{1}{r} & 0 \\ 0 & -\frac{1}{r} \end{pmatrix}$$

$$\Rightarrow \quad \boxed{\begin{array}{l} H \equiv -\frac{2}{r} \\ K \equiv \frac{1}{r^2} \end{array}}$$

Constant mean & Gauss curvature